Lagrangian-Eulerian method for two-dimensional hyperbolic conservation laws

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Objectives

Bibliographic review

Lagrangian-Eulerian scheme in one dimension

Some numerical tests with the Lagrangian-Eulerian scheme

Basic ideas for the extension of the Lagrangian-Eulerian scheme to two dimensions

Objectives

- Construct an improved Lagrangian-Eulerian conservative scheme on a triangular mesh to solve the hyperbolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} + \dfrac{\partial g(u)}{\partial y} = 0, \ (x, y, t) \in \Omega \times (0, T], \\ \qquad\qquad\qquad u(x, y, 0) = \eta(x, y), \ (x, y) \in \Omega. \end{cases}$$

- Advance in the formulation, construction and application of an improved Lagrangian-Eulerian conservative scheme on a triangular mesh for hyperbolic conservation laws with source term (non-homogeneous case).

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FVATM (A new finite volume approach for transport models and related applications with balancing source terms) Abreu, E., Lambert, W., Perez, J., Santo, A. (2017). Mathematics and Computers in Simulation.

FRSLES (A fast, robust, and simple Lagrangian–Eulerian solver for balance laws and applications) Abreu, E., Pérez, J. (2019). Computers and Mathematics with Applications.

Lagrangian-Eulerian scheme in one dimension

Scalar hyperbolic problem

It is considered the one-dimensional hyperbolic conservation law:

$$\frac{\partial u}{\partial t} + \frac{\partial H(u)}{\partial x} = 0, \; x \in [a, b], \; t \geq 0, \tag{1}$$

with initial condition given by $u(x, 0) = \eta(x), x \in [a, b]$.

In the divergent form, (1) reads [Dafermos]

$$\nabla_{t,x} \cdot \begin{pmatrix} u \\ H(u) \end{pmatrix} = 0. \tag{2}$$

Discretization

Let $t^n = \sum_{i=0}^{n-1} \Delta t^i$ be the evolution in time after n time steps Δt^i. We define the region (cell-centered finite-volume [**Pereira et al. 2000**]),

$$D_j^n = \{(x,t) : t^n \le t \le t^{n+1}, \sigma_j^n(t) \le x \le \sigma_{j+1}^n(t)\},$$

and the intervals $I_j^n = \left[x_{j-1/2}^n, x_{j+1/2}^n\right]$, $\overline{I}_j^{n+1} = \left[\overline{x}_{j-\frac{1}{2}}^{n+1}, \overline{x}_{j+\frac{1}{2}}^{n+1}\right]$.
Let U_j^n and $\overline{U}_j^{n+1}$ be numerical approximations of u defined by

$$U_j^n = \frac{1}{h} \int_{x_{j-\frac{1}{2}}^{n+1}}^{x_{j+\frac{1}{2}}^{n+1}} u(x, t^n) dx, \quad \overline{U}_j^{n+1} = \frac{1}{h_j^{n+1}} \int_{\overline{x}_{j-\frac{1}{2}}^{n+1}}^{\overline{x}_{j+\frac{1}{2}}^{n+1}} u(x, t^{n+1}) dx. \tag{3}$$

For all $x \in I_j^n$ and for all $t \in [t^n, t^{n+1})$, let the piecewise constant function $U(x,t) = U_j^n$ be the approximate solution of (1).

Lagrangian-Eulerian formulation

[Pereira et al. 2000]

Integrating equation (2) over the region D_j^n of the Fig. (1) and using the Divergence theorem, we have

$$
\begin{aligned}
0 &= \int_{D_j^n} \nabla_{t,x} \cdot \begin{pmatrix} u \\ H(u) \end{pmatrix} dxdt \\
&= \int_{\partial D_j^n} \begin{pmatrix} u \\ H(u) \end{pmatrix} \cdot \overrightarrow{n}\, d(\partial D_j^n).
\end{aligned} \tag{4}
$$

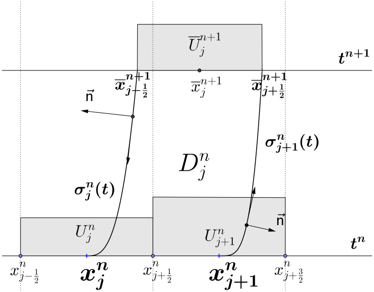
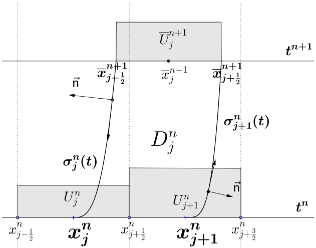


Figura: Integral Tube

Lagrangian-Eulerian formulation [Pereira et al. 2000]

Equation (4) implies first that

$$
\begin{aligned}
0 = &\int_{x_j^n}^{x_{j+1}^n} \begin{pmatrix} u(x,t^n) \\ H(u(x,t^n)) \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} dx \\
&+ \int_{\sigma_{j+1}^n} \begin{pmatrix} u \\ H(u) \end{pmatrix} \cdot \vec{n}\, d\sigma_{j+1}^n \\
&+ \int_{\sigma_j^n} \begin{pmatrix} u \\ H(u) \end{pmatrix} \cdot \vec{n}\, d\sigma_j^n \\
&+ \int_{\overline{x}_{j+\frac{1}{2}}^{n+1}}^{\overline{x}_{j-\frac{1}{2}}^{n+1}} \begin{pmatrix} u(x,t^{n+1}) \\ H(u(x,t^{n+1})) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} (-dx).
\end{aligned}
$$



This equality leads to the local conservation law:

$$
\int_{\overline{x}_{j-\frac{1}{2}}^{n+1}}^{\overline{x}_{j+\frac{1}{2}}^{n+1}} u(x,t^{n+1})dx = \int_{x_j^n}^{x_{j+1}^n} u(x,t^n)dx. \qquad (5)
$$

Lagrangian-Eulerian formulation [Pereira et al. 2000]

Since $\int_{\sigma_j^n} \begin{pmatrix} u \\ H(u) \end{pmatrix} \cdot \overrightarrow{n}\, d\sigma_j^n = 0$ then,

$$\begin{pmatrix} u \\ H(u) \end{pmatrix} \cdot \overrightarrow{n} = 0,\ \forall (x,t) \in \sigma_j^n. \tag{6}$$

Parameterizing σ_j^n as $r(t) = (t, \sigma_j^n(t)), t \in [t^n, t^{n+1}]$, we get the normal vector to the curve

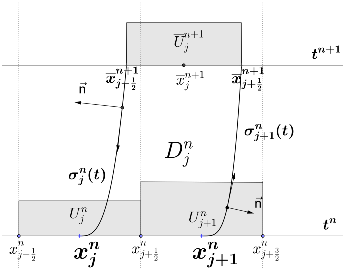
$$n(t) = \left(\frac{d\sigma_j^n(t)}{dt}, -1 \right).$$

Substituting $n(t)$ in (6) we get, for $t^n < t \leq t^{n+1}$,

$$\begin{cases} \dfrac{d\sigma_j^n(t)}{dt} = \dfrac{H(u)}{u}, \\ \sigma_j^n(t^n) \ = \ x_j^n. \end{cases} \tag{7}$$

Lagrangian-Eulerian formulation [Pereira et al. 2000]

From the local conservation law obtained, we get



$$\overline{U}_j^{n+1} = \frac{1}{h_j^{n+1}} \int_{\overline{x}_{j-\frac{1}{2}}^{n+1}}^{\overline{x}_{j+\frac{1}{2}}^{n+1}} u(x, t^{n+1})dx$$

$$= \frac{1}{h_j^{n+1}} \int_{x_j^n}^{x_{j+1}^n} u(x, t^n)dx$$

$$= \frac{1}{h_j^{n+1}} \left[h \left(\frac{1}{h} \int_{x_j^n}^{x_{j+\frac{1}{2}}^n} u(x, t^n)dx \right) + h \left(\frac{1}{h} \int_{x_{j+\frac{1}{2}}^n}^{x_{j+1}^n} u(x, t^n)dx \right) \right].$$

So,

$$\overline{U}_j^{n+1} = \frac{h}{2h_j^{n+1}} \left(U_j^n + U_{j+1}^n \right). \tag{8}$$

Lagrangian-Eulerian scheme **[Pérez 2015]**

From the IVP (7) that defines the integral curve, for each interval time $(t^n, t^{n+1}]$ we have

$$\sigma_j^n(t) = f_j^n(t - t^n) + x_j^n, j \in \mathbb{Z}, \tag{9}$$

where $f_j^n = \frac{H(U_j^n)}{U_j^n}$.

From Fig.(2) we have

$$U_j^{n+1} = \frac{1}{h}\left[c_{0j}\overline{U}_{j-1}^{n+1} + c_{1j}\overline{U}_j^{n+1} \right]. \tag{10}$$

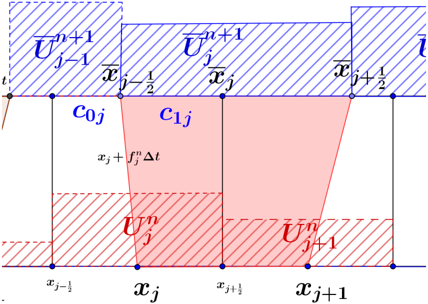


Figura: Approximated Integral Tube

Lagrangian-Eulerian scheme **[Pérez 2015]**

From the equation of the approximated integral curve (9), we have

$$\overline{x}^{n+1}_{j-\frac{1}{2}} = \sigma^n_j(t^{n+1}) = f^n_j(t^{n+1} - t^n) + x^n_j = x^n_{j-\frac{1}{2}} + c_{0j},$$

$$c_{0j} + c_{1j} = h.$$

These two equations lead to $c_{0j} = \frac{h}{2} + f_j^n \Delta t^n$ and $c_{1j} = \frac{h}{2} - f_j^n \Delta t^n$.

Finally, the Lagrangian-Eulerian scheme can be obtained from the algorithm,

$$\begin{cases} \overline{U}^{n+1}_j = \frac{h}{2h^{n+1}_j}\left(U^n_j + U^n_{j+1}\right), \\ \\ U^{n+1}_j = \frac{1}{h}\left[c_{0j}\overline{U}^{n+1}_{j-1} + c_{1j}\overline{U}^{n+1}_j\right]. \end{cases} \tag{11}$$

Lagrangian-Eulerian scheme. Linear case **[Pérez 2015]**

In this case $H(u) = au$. Therefore, $f_j^n = a$ and with this, $h_j^n = h$. So,

$$\overline{U}_j^{n+1} = \frac{1}{2}\left(U_j^n + U_{j+1}^n\right), \tag{12}$$

and

$$U_j^{n+1} = \frac{1}{h}\left[\left(\frac{h}{2} + a\Delta t^n\right)\overline{U}_{j-1}^{n+1} + \left(\frac{h}{2} - a\Delta t^n\right)\overline{U}_j^{n+1}\right]. \tag{13}$$

(12) and (13) lead to:

$$U_j^{n+1} = \frac{1}{4}(U_{j+1}^n + 2U_j^n + U_{j-1}^n) - \frac{a\Delta t^n}{2h}(U_{j+1}^n - U_{j-1}^n), \tag{14}$$

which corresponds to the Lagrangian-Eulerian scheme.

Lagrangian-Eulerian scheme. Nonlinear case **[Pérez 2015]**

The Lagrangian-Eulerian scheme for the nonlinear case is:

$$U_j^{n+1} = \frac{1}{4}(U_{j-1}^n + 2U_j^n + U_{j+1}^n) - \frac{\Delta t^n}{2h}\left(H(U_{j+1}^n) - H(U_{j-1}^n)\right). \quad (15)$$

The conservative form of (15) is:

$$U_j^{n+1} = U_j^n - \frac{\Delta t^n}{h}[F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n)], \quad (16)$$

where

$$F(U_j^n, U_{j+1}^n) = \frac{h}{4\Delta t^n}(U_j^n - U_{j+1}^n) + \frac{1}{2}(H(U_{j+1}^n) + H(U_j^n))$$

is called *Numerical flux*.

Properties

- The scheme is **locally conservative**.
- The numerical flux is **monotonous**: $F(v, w)$ is nondecreasing with respect to v and nonincreasing with respect to w if

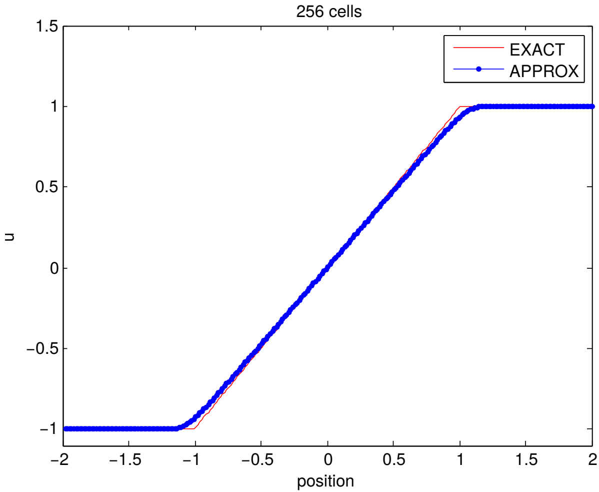
$$\frac{\Delta t}{h} \left| \max \left\{ H'(u) \right\} \right| \leq \frac{1}{2}.$$

- The numerical flux is **Lipschitz continuous** if $H \in C^1(K)$ with K a compact set of $\mathbb{R}$.
- The numerical flux is **consistent**: $F(s, s) = H(s)$.

Some numerical tests with the Lagrangian-Eulerian scheme

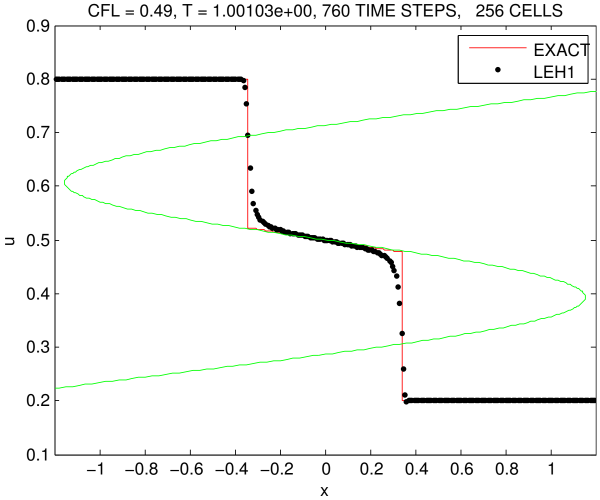
Numerical tests

Burgers Inviscid equation, $u_t + \left(\frac{u^2}{2}\right)_x = 0$, with initial condition $u_l = -1, x < 0,\ u_r = 1, x > 0.$



Numerical tests

Nonconvex flux, $u_t + (H(u))_x = 0$, $H(u) = 0{,}5(e^{-25(u-0{,}5)^2} + 8(u - 0{,}5)^2)$ with initial condition $u_l = 1, x < 0$, $u_r = 0, x > 0$.



Extension of the Lagrangian-Eulerian scheme to two dimensions

- Use square and triangular meshes.
- Put the numerical flux in the faces of the cells.
- Keep the properties of the one-dimensional scheme.

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Thank you